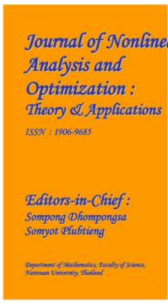
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 15, Issue. 1, No.1 : 2024
ISSN : **1906-9685**



IRRATIONALS AND CONTINUED FRACTIONS

R. Jayamari, Ph.D. Research Scholar, Post Graduate and Research Department of Mathematics
Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, India
Email: jayamari85@gmail.com
Dr. R. Sivaraman Associate Professor, Department of Mathematics Post Graduate and Research Department of Mathematics Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, India
Email: rsivaraman1729@yahoo.co.in

Abstract
The study of continued fractions has created a great deal of interest among mathematicians for several centuries. Various continued fractions have been developed through which we have gained new insights upon understanding the behaviour of numbers. In this paper, we will construct a general continued fraction for $\sqrt{k^2+4}$ and using that, we have obtained the rational approximations for some of the irrational numbers like $\sqrt{5}, \sqrt{13}, \sqrt{29}, \sqrt{53}, \ldots$

Keywords: Continued Fraction, Convergent, Irrationals, Approximations

1. Introduction

The study of irrational numbers has been done more than two millennia ago. Ever since, Pythagoreans discovered the irrational numbers, its structure and properties were studied extensively by mathematicians spanning several centuries to this day. The concept of continued fractions was popularized by the great Swiss mathematician Leonhard Euler, who was master of proving exciting equations covering all branches of mathematics. In this paper, we will derive the continued fraction expansion of the expression of the form $\sqrt{k^2+4}$ and using this result, we will arrive the rational approximations of three irrational numbers $\sqrt{5}$, $\sqrt{13}$, $\sqrt{29}$.

2.DEFINITIONS AND NOTATIONS

2.1 A finite continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} \qquad (1)$$

where $a_0, a_1, \ \ldots\ a_n$ are real numbers and $a_1, a_2, \ \ldots a_n$ are positive. The a_i are called the partial quotients of the continued fraction. If the partial quotients are all integers, then the continued fraction is simple. We use the notation $[a_0; a_1, \ \ldots a_n]$ to represent the continued fraction given in equation (1). When $n = 0$ we write $[a_0]$

2.2 For $1 \leq k \leq n$, the k^{th} convergent C_K of a continued fraction $[a_0; a_1, a_{2,} \ldots . a_n]$ is the continued fraction

$$C_k = [a_0; a_1, a_2, \ldots\ldots a_k]$$

We extend this definition to include $k = 0$ and so we set $C_0 = a_0$.

We now prove the following theorem.

3. Theorem 1

If k is a positive integer, then the continued fraction expansion for $\sqrt{k^2+4}$ is given by

$$\sqrt{k^2+4} = k + \cfrac{2}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cdots}}}}} \quad (2)$$

Proof: Consider, $\left(\frac{k-\sqrt{k^2+4}}{2}\right)\left(\frac{k+\sqrt{k^2+4}}{2}\right) = -1$

$$\left(\frac{k-\sqrt{k^2+4}}{2}\right) = \frac{-1}{\left(\frac{k+\sqrt{k^2+4}}{2}\right)} \quad (3)$$

$$= \frac{-1}{k-\left(\frac{k-\sqrt{k^2+4}}{2}\right)}$$

$$= \cfrac{-1}{k+\cfrac{1}{\left(\frac{k+\sqrt{k^2+4}}{2}\right)}} \quad \text{by (3)}$$

$$= \cfrac{-1}{k+\cfrac{1}{k-\left(\frac{k-\sqrt{k^2+4}}{2}\right)}}$$

$$= \cfrac{-1}{k+\cfrac{1}{k+\cfrac{1}{\left(\frac{k+\sqrt{k^2+4}}{2}\right)}}}$$

$$= \cfrac{-1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cdots}}}}}$$

$$\sqrt{k^2+4} = k + \cfrac{2}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cdots}}}}}$$

This proves (2) and hence completes the proof.

4. Theorem 2

For any positive integer k, the continued fraction $k + \cfrac{2}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cdots}}}}$ converges to $\sqrt{k^2+4}$.

Proof: Let $x = k + \cfrac{2}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k\cdots}}}}$

$$x - k = \cfrac{2}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k\cdots}}}}$$

$$\frac{x-k}{2} = \cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k\cdots}}}} \quad (4)$$

$$\frac{x-k}{2} = \cfrac{1}{k+\frac{x-k}{2}} \quad \text{by (4)}$$

$$\frac{x-k}{2} = \frac{2}{2k+x-k} = \frac{2}{x+k}$$

$$(x-k)(x+k) = 4$$

$$x^2 - k^2 = 4$$

$$x = \sqrt{k^2 + 4}$$

Thus the continued fraction $k + \frac{2}{k+\frac{1}{k+\frac{1}{k+\frac{1}{k\cdots}}}}$ converges to $\sqrt{k^2 + 4}$. This completes the proof.

We now present three corollaries as verification of Theorem 2.

4.1 Corollary 1

$\sqrt{5} = 2.2360679$

Proof: Let k=1. Then $\sqrt{k^2 + 4}$= $\sqrt{1^2 + 4}$= $\sqrt{5} = 2.2360679$

By (2) of Theorem 1,$\sqrt{1^2 + 4}$ = $\sqrt{5} = 1 + \frac{2}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}}}$

Extracting the consecutive convergents, we obtain

$$c_0 = 1$$
$$c_1 = 1 + \frac{2}{1} = 3$$
$$c_2 = 1 + \frac{2}{1+\frac{1}{1}} = 1 + \frac{2}{2} = 2$$
$$c_3 = 1 + \frac{2}{1+\frac{1}{1+\frac{1}{1}}} = 1 + \frac{2}{1+\frac{1}{2}} = 1 + \frac{4}{3} = \frac{7}{3} = 2.3333333$$
$$c_4 = 1 + \frac{2}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} = 1 + \frac{2}{1+\frac{2}{3}} = 1 + \frac{2}{\frac{5}{3}} = 1 + \frac{6}{5} = \frac{11}{5} = 2.2$$
$$c_5 = 1 + \frac{2}{1+\frac{3}{5}} = 1 + \frac{2}{\frac{8}{5}} = 1 + \frac{10}{8} = \frac{18}{8} = 2.25$$
$$c_6 = 1 + \frac{2}{1+\frac{5}{8}} = 1 + \frac{2}{\frac{13}{8}} = 1 + \frac{16}{13} = \frac{29}{13} = 2.230769$$

If we now list the convergents obtained above, we get

List ofconvergents c_n

***n*=0,2,4,6,8 (Lower bounds)**	***n*=1,3,5,7 (Upper bounds)**
$c_0 = 1$	$c_1 = 3$
$c_2 = 2$	$c_3 = 2.3333333$
$c_4 = 2.2$	$c_5 = 2.25$
$c_6 = 2.230769$	$c_7 = 2.2380952$
$c_8 = 2.235294$	$c_9 = 2.23636363$

Thus the above convergents forming lower and upper bounds respectively eventually converges to $\sqrt{5} = 2.2360679$. This verifies the result obtained in Theorem 2.

4.2 Corollary 2

$\sqrt{13} = 3.60555127$

Proof: Let k = 3. Then$\sqrt{k^2 + 4}$ = $\sqrt{3^2 + 4}$ = $\sqrt{13} = 3.60555127$

By (2) of Theorem 1,$\sqrt{3^2 + 4}$ = $\sqrt{13} = 3 + \frac{2}{3+\frac{1}{3+\frac{1}{3+\frac{1}{3+\frac{1}{3+\cdots}}}}}$

$$c_0 = 3$$

$c_1 = 3 + \frac{2}{3} = \frac{11}{3} = 3.6666666$

$c_2 = 3 + \frac{2}{3+\frac{1}{3}} = 3 + \frac{2}{\frac{10}{3}} = 3 + \frac{6}{10} = \frac{36}{10} = 3.6$

$c_3 = 3 + \frac{2}{3+\frac{1}{3+\frac{1}{3}}} = 3 + \frac{2}{3+\frac{3}{10}} = 3 + \frac{2}{\frac{33}{10}} = 3 + \frac{20}{33} = \frac{119}{33} = 3.606060$

$c_4 = 3 + \frac{2}{3+\frac{1}{3+\frac{1}{3+\frac{1}{3}}}} = 3 + \frac{2}{3+\frac{10}{33}} = 3 + \frac{2}{\frac{109}{33}} = 3 + \frac{66}{109}$

$= \frac{393}{109} = 3.6055045$

If we now list the convergents obtained above, we get

List of convergents c_n

n=0,2,4,6 (Lower bounds)	*n*=1,3,5,7 (Upper bounds)
$c_0 = 3$	$c_1 = 3.6666666$
$c_2 = 3.6$	$c_3 = 3.60606060$
$c_4 = 3.60550458$	$c_5 = 3.60555555$
$c_6 = 3.6055508$	$c_7 = 3.60555131$

Thus the above convergents forming lower and upper bounds respectively eventually converges to $\sqrt{13} = 3.60555127$. This verifies the result obtained in Theorem 2.

4.3 Corollary 3

$\sqrt{29} = 5.38516480$

Proof: Let $k = 5$. Then$\sqrt{k^2+4}$= $\sqrt{5^2+4}$= $\sqrt{29} = 5.38516480$

By (2) of Theorem 1,$\sqrt{5^2+4}$ = $\sqrt{29} = 5 + \frac{2}{5+\frac{1}{5+\frac{1}{5+\frac{1}{5+\frac{1}{5+\cdots}}}}}$

$c_0 = 5$

$c_1 = 5 + \frac{2}{5} = 5.4$

$c_2 = 5 + \frac{2}{5+\frac{1}{5}} = 5 + \frac{10}{26} = 5.3846153$

$c_3 = 5 + \frac{2}{5+\frac{1}{5+\frac{1}{5}}} = 5 + \frac{2}{5+\frac{5}{26}} = 5 + \frac{52}{135} = 5.3851851$

$c_4 = 5 + \frac{2}{5+\frac{26}{135}} = 5 + \frac{270}{701} = 5.385164051$

$c_5 = 5 + \frac{2}{5+\frac{135}{701}} = 5 + \frac{1402}{3640} = 5.3851648351$

$c_6 = 5 + \frac{2}{5+\frac{701}{3640}} = 5 + \frac{7280}{18901} = 5.3851648060$

If we now list the convergents obtained above, we get

List of convergents c_n

n=0,2,4,6 (Lower bounds)	*n*=1,3,5,7 (Upper bounds)
$c_0 = 5$	$c_1 = 5.4$

$c_2 = 5.3846153$	$c_3 = 5.3851851$
$c_4 = 5.385164051$	$c_5 = 5.3851648351$
$c_6 = 5.3851648060$	$c_7 = 5.3851648071$

Thus the above convergents forming lower and upper bounds respectively eventually converges to $\sqrt{29} = 5.38516480$. This verifies the result obtained in Theorem 2.

5. Conclusion

In this paper, in Theorem 1, we have constructed a continued fraction expansion for a quadratic irrational surd $\sqrt{k^2+4}$ as in (2). In doing so, we can obtain rational approximations of several irrational numbers.In Theorem 2, we proved that the rational approximations which are values corresponding to successive convergents of the continued fraction in (2), indeed converges to $\sqrt{k^2+4}$.The rational approximations of numbers $\sqrt{k^2+4}$ for k = 1, 3, 5 were obtained in corollaries 1 to 3. By generalizing these kind of computations, we can determine the rational approximations of any irrational numbers of the form $\sqrt{n}$ when n is not a perfect square.

REFERENCES

[1] N. Calkin, H.S. Wilf, Recounting the Rationals, American Mathematical Monthly 107 (4) (2000) 360-363

[2] R. Sivaraman, Triangle of Triangular Numbers, International Journal of Mathematics and Computer Research, Volume 9, Issue 10, October 2021, pp. 2390 – 2394.

[3] R. Sivaraman, Recognizing Ramanujan's House Number Puzzle, German International Journal of Modern Science, 22, November 2021, pp. 25 – 27

[4] R. Sivaraman, J. Suganthi, A. Dinesh Kumar, P.N. Vijayakumar, R. Sengothai, On Solving an Amusing Puzzle, Specialusis Ugdymas/Special Education, Vol 1, No. 43, 2022, 643 – 647.

[5] A. Dinesh Kumar, R. Sivaraman, On Some Properties of Fabulous Fraction Tree, Mathematics and Statistics, Vol. 10, No. 3, (2022), pp. 477 – 485.

[6] R. Sengothai, R. Sivaraman, Solving Diophantine Equations using Bronze Ratio, Journal of Algebraic Statistics, Volume 13, No. 3, 2022, 812 – 814.

[7] P.N.Vijayakumar, R. Sivaraman, On Solving Euler's Quadratic Diophantine Equation, Journal of Algebraic Statistics, Volume 13, No. 3, 2022, 815 – 817.

[8] R. Sivaraman, Generalized Lucas, Fibonacci Sequences and Matrices, Purakala, Volume 31, Issue 18, April 2020, pp. 509 – 515.

[9] R. Sivaraman, J. Suganthi, P.N. Vijayakumar, R. Sengothai, Generalized Pascal's Triangle and its Properties, NeuroQuantology, Vol. 22, No. 5, 2022, 729 – 732.

[10] A. Dinesh Kumar, R. Sivaraman, Asymptotic Behavior of Limiting Ratios of Generalized Recurrence Relations, Journal of Algebraic Statistics, Volume 13, No. 2, 2022, 11 – 19.

[11] A. Dinesh Kumar, R. Sivaraman, Analysis of Limiting Ratios of Special Sequences, Mathematics and Statistics, Vol. 10, No. 4, (2022), pp. 825 – 832

[12] Andreescu, T., D. Andrica, and I. Cucurezeanu, An introduction to Diophantine equations: A problem-based approach, Birkhäuser Verlag, New York, 2010.